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Higher order periodic solutions of coupled ϕ^4 and ϕ^6 models

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Abstract

We obtain several higher order periodic solutions of (i) a coupled symmetric ϕ^4 model in an external field, (ii) a coupled asymmetric ϕ^4 model, (iii) a coupled symmetric–asymmetric ϕ^4 model and (iv) a coupled ϕ^6 model in terms of Lamé polynomials and obtain the corresponding hyperbolic solutions in the appropriate limit. All these solutions are unusual in the sense that while they are the solutions of the coupled problems, they are *not* the solutions of the corresponding uncoupled problems. Possible physical applications of these solutions include periodic domain walls in magnetic and structural phase transitions as well as in field theory.

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1. Introduction

Coupled double well (ϕ^4) and triple well (ϕ^6) one-dimensional potentials are prevalent in both condensed matter physics and field theory. Few examples of ϕ^4 type double well models, which are of current interest, include spin configurations, domain walls and magnetic phase transitions in multiferroic materials [1, 2] and ω phase transition in various elements (e.g. Ti and Zr) and alloys [3]. Similarly, coupled ϕ^6 models are of interest in the context of structural phase transitions [4, 5] as well as scalar field theories [6–8]. In two recent publications [9, 10], we obtained a large number of periodic solutions, in terms of the Lamé polynomials of order 1 [11, 12], for (i) a coupled symmetric ϕ^4 model in an external field and (ii) an asymmetric coupled ϕ^4 model, both models with a biquadratic coupling. Further, in another publication [13], we obtained a large number of periodic solutions of a coupled ϕ^6 model. All those solutions had the feature that in the uncoupled limit, they reduce to the well-known solutions of the uncoupled symmetric or asymmetric double well problem (or the triple well problem) as the case may be.

The purpose of this paper is to point out that all these coupled models have, in addition, truly novel solutions, in terms of the higher order Lamé polynomials, which only exist due to the presence of the coupling between the two fields. In particular, while the Lamé polynomials of order 2 are solutions of the coupled ϕ^4 problems, they are *not* the solutions of the decoupled ϕ^4 problem. Similarly, while Lamé polynomials of order 1 and 2 are the solutions of the coupled ϕ^6 problem, they are *not* the solutions of the decoupled ϕ^6 problem. For completeness, we also consider a coupled asymmetric–symmetric ϕ^4 model (which corresponds to a first-order transition in one field and a second-order transition in the other) and show that Lamé polynomials of order 2 are the solutions of this coupled problem, even though they are *not* the solutions of the uncoupled problem. This model is relevant for certain martensitic transformations in elements [14, 15]. The significance of these solutions is that in the relevant physical systems with two coupled variables (e.g. magnetization and strain in magnetoelastic materials) they represent periodic domain walls in both variables. Such periodic array of domain walls would not exist if there was no (e.g. magnetoelastic) coupling. We emphasize that in this paper we only give those solutions of the coupled problems which are *not* the solutions of the uncoupled problems. Further, none of these solutions, in terms of higher order Lamé polynomials, are contained in our recent papers on the coupled problems [9, 10, 13].

The paper is organized as follows. In section 2, we provide the novel periodic as well as the corresponding hyperbolic solutions for a coupled symmetric ϕ^4 model with an explicit biquadratic coupling in the presence of an external field (with an additional linear–quadratic coupling) [9]. Note that the symmetric ϕ^4 model, in the decoupled limit, corresponds to a second-order transition in both the fields. We show that while the solutions of the uncoupled ϕ^4 problem are the Lamé polynomials of order 1, (i.e. sn, cn, dn), for the coupled problem, not only the Lamé polynomials of order 1 [9], but even the Lamé polynomials of order 2 are solutions of the coupled field equations. In section 3, we provide similar novel periodic solutions in terms of Lamé polynomials of order 2 for a coupled asymmetric ϕ^4 model, which corresponds to a first-order transition in both the fields [10]. In section 4, we consider similar higher order solutions for a symmetric–asymmetric ϕ^4 model. In section 5, we obtain the Lamé polynomial solutions of order 1 and 2 of a coupled ϕ^6 model even though they are *not* the solutions of the uncoupled ϕ^6 model. Finally, we conclude in section 6 with summary and possible extensions. In view of the shortage of space, we only write down the important steps, omitting the details which can be found elsewhere [16, 17].

2. Coupled symmetric ϕ^4 model in an external field

This model and the periodic domain wall solutions obtained here are relevant to spin configurations and magnetic phase transitions in multiferroic materials. In [9] we had considered the following potential, with a *biquadratic* coupling between the two fields and in an external magnetic field (H_z)

$$V = \alpha_1\phi^2 + \beta_1\phi^4 + \alpha_2\psi^2 + \beta_2\psi^4 + \gamma\phi^2\psi^2 - H_z[\rho_1\phi + \rho_2\phi^3 + \rho_3\phi\psi^2], \quad (1)$$

where α_i, β_i, γ and ρ_i are material (or system) dependent parameters. For $\alpha_1 < 0, \alpha_2 < 0$ and $\beta_1 > 0, \beta_2 > 0$, this model corresponds to second-order transitions in both fields ϕ and ψ. The corresponding (static) equations of motion are

$$\frac{d^2\phi}{dx^2} = 2\alpha_1\phi + 4\beta_1\phi^3 + 2\gamma\phi\psi^2 - H_z[\rho_1 + 3\rho_2\phi^2 + \rho_3\psi^2], \quad (2)$$

$$\frac{d^2\psi}{dx^2} = 2\alpha_2\psi + 4\beta_2\psi^3 + 2\gamma\phi^2\psi - 2H_z\rho_3\phi\psi. \quad (3)$$

These coupled set of equations admit several novel periodic solutions (i.e. Lamé polynomials of order 2), which we now discuss one-by-one systematically.

2.1. Solution I

It is not difficult to show that

$$\phi = F + A\,\mathrm{sn}^2[D(x+x_0), m], \qquad \psi = G + B\,\mathrm{sn}^2[D(x+x_0), m] \tag{4}$$

is an exact solution to coupled field equations (2) and (3) provided the following eight field equations are satisfied:

$$2\alpha_1 F + 4\beta_1 F^3 + 2\gamma F G^2 - H_z\rho_1 - 3H_z\rho_2 F^2 - H_z\rho_3 G^2 = 2AD^2, \tag{5}$$

$$2\alpha_1 A + 12\beta_1 F^2 A + 4\gamma BFG + 2\gamma AG^2 - 6H_z\rho_2 AF - 2H_z\rho_3 BG = -4(1+m)AD^2, \tag{6}$$

$$12\beta_1 FA^2 + 2\gamma FB^2 + 4\gamma ABG - 3H_z\rho_2 A^2 - H_z\rho_3 B^2 = 6AmD^2, \tag{7}$$

$$2\beta_1 A^2 + \gamma B^2 = 0, \tag{8}$$

$$2\alpha_2 G + 4\beta_2 G^3 + 2\gamma GF^2 - 2H_z\rho_3 GF = 2BD^2, \tag{9}$$

$$2\alpha_2 B + 12\beta_2 G^2 B + 4\gamma AFG + 2\gamma BF^2 - 2H_z\rho_3(BF + AG) = -4(1+m)BD^2, \tag{10}$$

$$12\beta_2 GB^2 + 2\gamma GA^2 + 4\gamma ABF - 2H_z\rho_3 AB = 6mBD^2, \tag{11}$$

$$2\beta_2 B^2 + \gamma A^2 = 0. \tag{12}$$

Here A and B denote the amplitudes of the 'pulse lattice', F and G are constants, D is an inverse characteristic length and x_0 is the (arbitrary) location of the pulse; m denotes the modulus of the Jacobi elliptic function $\mathrm{sn}\,(x, k)$. Five of these equations determine the five unknowns A, B, D, F, G while the other three equations give three constraints between the nine parameters $\alpha_{1,2}, \beta_{1,2}, \gamma, H_z, \rho_1, \rho_2, \rho_3$. In particular, from equations (8) and (12) it follows that

$$\gamma < 0, \qquad |\gamma|^2 = 4\beta_1\beta_2, \qquad \sqrt{\beta_1}A^2 = \sqrt{\beta_2}B^2. \tag{13}$$

Few comments are in order at this stage.

(1) From equation (9) it follows that no solution of form (4) exists in case $G = 0$. Thus no solutions exist with $\psi = B\,\mathrm{sn}^2[D(x+x_0), m]$ irrespective of the value of F. In fact one can also show that no solution exists in case $B = -G$ or if $B = -mG$ unless $m = 1$. In other words, even the solutions of the form $\psi = G\mathrm{cn}^2[D(x+x_0), m]$ or $\psi = G\,\mathrm{dn}^2[D(x+x_0), m]$ do not exist, no matter what F is, except when $m = 1$.

(2) In the special case of $H_z = 0$, the field equations (5)–(12) are completely symmetrical in ϕ and ψ. It is easily shown that in this case, solution (4) does not exist.

Solution at $m = 1$. In the special case of $m = 1$, the solution (4) goes over to the hyperbolic nontopological soliton solution

$$\phi = F + A\tanh^2[D(x+x_0)], \qquad \psi = G + B\tanh^2[D(x+x_0)], \tag{14}$$

provided the field equations (5)–(12) with $m = 1$ are satisfied. This hyperbolic soliton solution takes particularly simple form in two cases which we mention one by one.

(i) $F = 0, G = -B$. In this case, the nontopological soliton solution (14) takes the simpler form

$$\phi = A\tanh^2[D(x+x_0)], \qquad \psi = B\,\mathrm{sech}^2[D(x+x_0)]. \tag{15}$$

By analyzing equations (5)–(12) it is easily shown that such a solution exists provided $\gamma < 0, \alpha_2 < 0, \rho_3 < 0$.

(ii) $F = -A, G = -B$. In this limit the nontopological soliton solution (14) takes the simpler form

$$\phi = A\,\mathrm{sech}^2[D(x+x_0)], \qquad \psi = B\,\mathrm{sech}^2[D(x+x_0)] \tag{16}$$

provided

$$D^2 = \frac{\alpha_1}{2}, \qquad A = \frac{3\alpha_1}{2H_z\rho_3}, \qquad 3\rho_2 A^2 = (2A^2 - B^2)\rho_3. \tag{17}$$

2.2. *Solution II*

It is not difficult to show that

$$\phi = F + A\,\mathrm{sn}^2[D(x+x_0), m], \qquad \psi = B\,\mathrm{sn}[D(x+x_0), m]\mathrm{cn}[D(x+x_0), m] \tag{18}$$

is an exact solution to coupled field equations (2) and (3) provided the following seven field equations are satisfied:

$$2\alpha_1 F + 4\beta_1 F^3 - H_z\rho_1 - 3H_z\rho_2 F^2 = 2AD^2, \tag{19}$$

$$2\alpha_1 A + 12\beta_1 F^2 A + 2\gamma F B^2 - 6H_z\rho_2 AF - H_z\rho_3 B^2 = -4(1+m)AD^2, \tag{20}$$

$$12\beta_1 F A^2 + 2\gamma B^2(A - F) - 3H_z\rho_2 A^2 + H_z\rho_3 B^2 = 6AmD^2, \tag{21}$$

$$2\beta_1 A^2 - \gamma B^2 = 0, \tag{22}$$

$$2\alpha_2 + 2\gamma F^2 - 2H_z\rho_3 F = -(4+m)D^2, \tag{23}$$

$$4\beta_2 B^2 + 4\gamma AF - 2H_z\rho_3 A = 6mD^2, \tag{24}$$

$$2\beta_2 B^2 - \gamma A^2 = 0. \tag{25}$$

Four of these equations determine the four unknowns A, B, D, F while the other three equations give three constraints between the nine parameters $\alpha_{1,2}, \beta_{1,2}, \gamma, H_z, \rho_1, \rho_2, \rho_3$. In particular, from equations (22) and (25) it follows that

$$\gamma > 0, \qquad \gamma^2 = 4\beta_1\beta_2, \qquad \sqrt{\beta_1}A^2 = \sqrt{\beta_2}B^2. \tag{26}$$

Solution at $m = 1$. In the special case of $m = 1$, the solution (18) goes over to the hyperbolic nontopological soliton solution

$$\phi = F + A\tanh^2[D(x+x_0)], \qquad \psi = B\tanh[D(x+x_0)]\mathrm{sech}[D(x+x_0)] \tag{27}$$

provided the field equations (19)–(25) with $m = 1$ are satisfied.

Special case $H_z = 0$. In the special case of $H_z = 0$, the field equations (2) and (3) are symmetrical in ϕ and ψ. In this case, equations (19)–(25) take rather simple form. In particular, in case $H_z = 0$, it is easily shown that the solution (18) exists provided

$$\gamma = 2\beta_1 = 2\beta_2, \qquad A^2 = B^2, \tag{28}$$

while the remaining equations take the simpler form

$$3mD^2 = (1+2x)\gamma A^2, \tag{29}$$

$$D^2 = \alpha_1 x + \gamma A^2 x^3, \tag{30}$$

$$-2(1+m)D^2 = \alpha_1 + x(1+3x)\gamma A^2, \tag{31}$$

$$-(4+m)D^2 = 2\alpha_2 + 2x^2\gamma A^2, \tag{32}$$

where $x = F/A$. On solving these equations, one finds that the only acceptable solution is given by

$$3mx = -(1+m) + \sqrt{1-m+m^2}, \tag{33}$$

using which one can then easily express D^2, α_2 and γA^2 in terms of α_1.

In particular, at $H_z = 0$ and $m = 1$, the solution (27) exists provided relations (28) are satisfied and further

$$F = -\frac{A}{3}, \qquad \alpha_1, \alpha_2 < 0, \qquad \gamma A^2 = 3D^2 = \frac{3|\alpha_1|}{2}, \qquad |\alpha_2| = \frac{7}{8}|\alpha_1|. \tag{34}$$

It is not difficult to show that unlike the solution (18), the solution

$$\phi = B\,\mathrm{sn}[D(x+x_0), m]\mathrm{cn}[D(x+x_0), m], \qquad \psi = F + A\,\mathrm{sn}^2[D(x+x_0), m], \tag{35}$$

exists only if $H_z = 0$. But since at $H_z = 0$, the field equations (2) and (3) are symmetrical in ϕ and ψ, hence (35) is a solution to field equations (2) and (3) provided equations (28)–(34) (with suitable change of parameters) are satisfied.

2.3. Solution III

It is not difficult to show that

$$\phi = F + A\,\mathrm{sn}^2[D(x+x_0), m], \qquad \psi = B\,\mathrm{sn}[D(x+x_0), m]\,\mathrm{dn}[D(x+x_0), m] \tag{36}$$

is an exact solution to coupled field equations (2) and (3) provided seven field equations similar to those for the solution (18) are satisfied. One can show that this solution exists provided

$$\gamma > 0, \qquad \gamma^2 = 4\beta_1\beta_2, \qquad \sqrt{\beta_1}A^2 = m\sqrt{\beta_2}B^2. \tag{37}$$

Solution at $m = 1$. In the special case of $m = 1$, the solution (36) goes over to the hyperbolic nontopological soliton solution (27).

Special case $H_z = 0$. In the special case of $H_z = 0$, the field equations (2) and (3) are symmetrical in ϕ and ψ. It is easily shown that the solution (36) exists provided equation (28) is satisfied and further $x = F/A$ is again given by equation (33) using which one can then easily express D^2, α_2 and γA^2 in terms of α_1. At $H_z = 0$ and $m = 1$, of course the solution goes over to the solution (27), which exists provided relations (28)–(34) are satisfied.

It is not difficult to show that unlike the solution (36), the solution

$$\phi = B\,\mathrm{sn}[D(x+x_0), m]\,\mathrm{dn}[D(x+x_0), m], \qquad \psi = F + A\,\mathrm{sn}^2[D(x+x_0), m], \tag{38}$$

exists only if $H_z = 0$. But since at $H_z = 0$, the field equations (2) and (3) are symmetrical in ϕ and ψ, hence (38) is a solution to field equations (2) and (3).

2.4. Solution IV

It is not difficult to show that

$$\phi = F + A\,\mathrm{sn}^2[D(x+x_0), m], \qquad \psi = B\,\mathrm{cn}[D(x+x_0), m]\,\mathrm{dn}[D(x+x_0), m], \tag{39}$$

is an exact solution to coupled field equations (2) and (3) provided seven field equations similar to those for the solution (18) are satisfied. One can show that such a solution exists provided

$$\gamma < 0, \qquad |\gamma|^2 = 4m\beta_1\beta_2, \qquad \sqrt{\beta_1}A^2 = \sqrt{m\beta_2}B^2. \tag{40}$$

Solution at $m = 1$. In the special case of $m = 1$, the solution (39) goes over to the hyperbolic nontopological soliton solution

$$\phi = F + A\tanh^2[D(x+x_0)], \qquad \psi = B\,\mathrm{sech}^2[D(x+x_0)], \tag{41}$$

which is essentially the solution (14) with $B = -G$.

Special case $H_z = 0$. In the special case of $H_z = 0$, the field equations (2) and (3) are symmetrical in ϕ and ψ. In this case, at least at $m = 1$, one can show that there is no solution to these equations. Of course this is expected since we know from the discussion of solution (4) that at $m = 1$ and $H_z = 0$, solution (41) does not exist.

3. Coupled asymmetric ϕ^4 model

Recently we had also considered a coupled asymmetric ϕ^4 model [10], which in the uncoupled limit corresponds to a first-order transition in both the fields, and had obtained periodic solutions in terms of the Lamé polynomials of order 1. The purpose of this section is to show that the Lamé polynomials of order 2 also constitute exact solutions of the same model, even though they are *not* the solutions of the asymmetric uncoupled model, thereby giving us genuinely nontrivial solutions of the model.

This model and the periodic domain walls solutions given below are relevant to structural transitions in certain materials. The potential that we considered in [10] is given by ($\beta_1 > 0, \beta_2 > 0$)

$$V = \alpha_1\phi^2 + \delta_1\phi^3 + \beta_1\phi^4 + \alpha_2\psi^2 + \delta_2\psi^3 + \beta_2\psi^4 + \gamma\phi^2\psi^2 + \eta\phi\psi^2, \tag{42}$$

where $\alpha_i, \delta_i, \beta_i, \gamma$ and η are material (or system) dependent parameters. Note that we have changed the notation slightly from that followed in [10], in order to be in conformity with the notation in the previous section. The (static) equations of motion which follow from here are

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = 2\alpha_1\phi + 3\delta_1\phi^2 + 4\beta_1\phi^3 + 2\gamma\phi\psi^2 + \eta\psi^2, \tag{43}$$

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = 2\alpha_2\psi + 3\delta_2\psi^2 + 4\beta_2\psi^3 + 2\gamma\phi^2\psi + 2\eta\psi\phi. \tag{44}$$

Observe that as long as $\eta \neq 0$, the two field equations are asymmetric in ϕ and ψ. We shall consider solutions of these coupled field equations in case $\alpha_i \neq 0, \delta_i \neq 0, \beta_i > 0$, as only then the model corresponds to a first-order transition in both the fields.

There is only one solution in this case. In particular

$$\phi = F + A\,\mathrm{sn}^2[D(x + x_0), m], \qquad \psi = G + B\,\mathrm{sn}^2[D(x + x_0), m], \tag{45}$$

is an exact solution to coupled field equations (43) and (44) provided eight field equations, similar to those for the solution (4), are satisfied. Five of these equations determine the five unknowns A, B, D, F, G, while the other three equations give three constraints between the eight parameters $\alpha_{1,2}, \delta_{1,2}, \beta_{1,2}, \gamma, \eta$. We find that this solution exists only if

$$\gamma < 0, \qquad |\gamma|^2 = 4\beta_1\beta_2, \qquad \sqrt{\beta_1}A^2 = \sqrt{\beta_2}B^2. \tag{46}$$

We also find that there is no solution in case either $G = 0$ or $-B$ or $-B/m$ so long as $m \neq 1$.

Solution at $m = 1$. In the special case of $m = 1$, the solution (45) goes over to the hyperbolic nontopological soliton solution

$$\phi = F + A\tanh^2[D(x + x_0)], \qquad \psi = G + B\tanh^2[D(x + x_0)]. \tag{47}$$

This hyperbolic soliton solution takes a particularly simple form in two cases which we mention one by one.

(i) $F = 0, G = -B$. In this limit the nontopological soliton solution (47) takes the simpler form

$$\phi = A\tanh^2[D(x + x_0)], \qquad \psi = B\,\mathrm{sech}^2[D(x + x_0)]. \tag{48}$$

Such a solution exists provided $\gamma < 0, \alpha_2 < 0$.

(ii) $F = -A, G = -B$. In this limit the nontopological soliton solution (47) takes the simpler form

$$\phi = A\,\mathrm{sech}^2[D(x+x_0)], \qquad \psi = B\,\mathrm{sech}^2[D(x+x_0)], \tag{49}$$

provided

$$\alpha_1 = \alpha_2 > 0, \qquad 2D^2 = \alpha_1, \qquad -6AD^2 = 3\delta_1 A^2 + \eta B^2, \qquad -6D^2 = 3\delta_2 B + 2\eta A. \tag{50}$$

It turns out that as long as $\delta_2 \neq 0$, no other Lamé polynomials of order 2 form a solution of field equations (43) and (44).

4. Asymmetric-symmetric ϕ^4 model

In the last section, we considered solutions in cases where both δ_1 and δ_2 are nonzero, i.e. solutions of the asymmetric ϕ^4 problem such that in both ψ and ϕ fields one has a first-order phase transition. In this section, we consider the case when $\delta_1 \neq 0$ while $\delta_2 = 0$. This corresponds to having a first-order transition in ϕ and a second-order transition in ψ. There are interesting physical situations such as a face-centered cubic to a hexagonal close packed (FCC-HCP) reconstructive structural transition [14] and the martensitic transition in cobalt [15] where this model is relevant. The solutions obtained below correspond to periodic domain walls between FCC and HCP crystal structures. Therefore, in this section we consider such a coupled model and obtain various solutions of this coupled model in terms of Lamé polynomials of order 2 and their hyperbolic limit.

It is worth pointing out that as in the coupled symmetric and asymmetric ϕ^4 cases, in the symmetric–asymmetric ϕ^4 model too, there are solutions in terms of Lamé polynomials of order 1 which can be easily written down [16]. However, we do not discuss these here.

The potential we consider is given by ($\beta_1 > 0, \beta_2 > 0$)

$$V = \alpha_1\phi^2 - \delta_1\phi^3 + \beta_1\phi^4 + \eta\phi\psi^2 + \gamma\phi^2\psi^2 + \alpha_2\psi^2 + \beta_2\psi^4, \tag{51}$$

where $\alpha_{1,2}, \beta_{1,2}, \delta_1, \eta, \gamma$ are system-dependent parameters. The static field equations that follow from here are

$$\phi_{xx} = 2\alpha_1\phi - 3\delta_1\phi^2 + 4\beta_1\phi^3 + \eta\psi^2 + 2\gamma\phi\psi^2, \tag{52}$$

$$\psi_{xx} = 2\alpha_2\psi + 4\beta_2\psi^3 + 2\eta\phi\psi + 2\gamma\phi^2\psi. \tag{53}$$

Observe that as long as $\eta \neq 0$, the two field equations are asymmetric and hence kink-pulse and pulse-kink solitons would be distinct.

For the uncoupled model ($\eta = \gamma = 0$), it is easy to show that the potential in ϕ corresponds to a first-order transition while that in ψ corresponds to a second-order transition.

We now show that the field equations (52) and (53) admit Lamé polynomial solutions of order 2, which are *not* the solutions of the uncoupled problem.

4.1. Solution I

It is not difficult to show that

$$\phi = F + A\,\mathrm{sn}^2[D(x+x_0), m], \qquad \psi = G + B\,\mathrm{sn}^2[D(x+x_0), m] \tag{54}$$

is an exact solution to coupled field equations (52) and (53) provided the field equations similar to those for the solution (4) are satisfied.

Solution at $m = 1$. In the special case of $m = 1$, the solution (54) goes over to the hyperbolic nontopological soliton solution (14). This hyperbolic soliton solution takes a particularly simple form in two cases which we mention one by one.

(i) $F = 0, G = -B$. In this limit the nontopological soliton solution (14) takes the simpler form

$$\phi = A\tanh^2[D(x+x_0)], \qquad \psi = B\,\mathrm{sech}^2[D(x+x_0)]. \tag{55}$$

It is easily shown that such a solution exists provided $\gamma < 0, \alpha_2 < 0$.

(ii) $F = -A, G = -B$. In this limit the nontopological soliton solution (14) takes the simpler form

$$\phi = A\,\mathrm{sech}^2[D(x+x_0)], \qquad \psi = B\,\mathrm{sech}^2[D(x+x_0)]. \tag{56}$$

Such a solution exists provided

$$\alpha_1 = \alpha_2 > 0, \qquad 2D^2 = \alpha_1, \qquad A = -\frac{3\alpha_1}{4\eta}, \qquad B^2 = \frac{9\alpha_1^2}{16\eta^3}4\eta - 3\delta_1. \tag{57}$$

4.2. Solution II

Unlike the previous section, it turns out that in view of $\delta_2 = 0$, now three more Lamé polynomial solutions of order 2 are allowed which we present one by one.

It is not difficult to show that

$$\phi = F + A\,\mathrm{sn}^2[D(x+x_0), m], \qquad \psi = B\,\mathrm{sn}[D(x+x_0), m]\mathrm{cn}[D(x+x_0), m] \tag{58}$$

is an exact solution to coupled field equations (52) and (53) provided seven field equations similar to those for solution (18) are satisfied. In particular, such a solution exists provided

$$\gamma > 0, \qquad \gamma^2 = 4\beta_1\beta_2, \qquad \sqrt{\beta_1}A^2 = \sqrt{\beta_2}B^2. \tag{59}$$

Further, no solution exists in case $F = 0$ or $-A$ or $-A/m$ unless $m = 1$.

Solution at $m = 1$. In the special case of $m = 1$, the solution (58) goes over to the hyperbolic nontopological soliton solution

$$\phi = F + A\tanh^2[D(x+x_0)], \qquad \psi = B\tanh[D(x+x_0)]\mathrm{sech}[D(x+x_0)]. \tag{60}$$

It is not difficult to show that unlike the solution (58), the solution

$$\phi = B\,\mathrm{sn}[D(x+x_0), m]\mathrm{cn}[D(x+x_0), m], \qquad \psi = F + A\,\mathrm{sn}^2[D(x+x_0), m], \tag{61}$$

does not exist as long as δ_1 and η are nonzero.

4.3. Solution III

Another allowed solution is

$$\phi = F + A\,\mathrm{sn}^2[D(x+x_0), m], \qquad \psi = B\,\mathrm{sn}[D(x+x_0), m]\,\mathrm{dn}[D(x+x_0), m], \tag{62}$$

provided seven field equations similar to those for the solution (18) are satisfied. In particular, one can show that solution (62) exists provided

$$\gamma > 0, \qquad \gamma^2 = 4\beta_1\beta_2, \qquad \sqrt{\beta_1}A^2 = m\sqrt{\beta_2}B^2. \tag{63}$$

Solution at $m = 1$. In the special case of $m = 1$, the solution (62) goes over to the hyperbolic nontopological soliton solution (60).

4.4. Solution IV

Finally, another allowed solution is

$$\phi = F + A\,\mathrm{sn}^2[D(x+x_0), m], \qquad \psi = B\,\mathrm{cn}[D(x+x_0), m]\,\mathrm{dn}[D(x+x_0), m], \tag{64}$$

provided coupled equations similar to those for the solution (18) are satisfied.

In particular, one can show that such a solution exists only if

$$\gamma < 0, \qquad |\gamma|^2 = 4m\beta_1\beta_2, \qquad \sqrt{\beta_1}A^2 = \sqrt{m\beta_2}B^2. \tag{65}$$

Solution at $m = 1$. In the special case of $m = 1$, the solution (64) goes over to the hyperbolic nontopological soliton solution

$$\phi = F + A\tanh^2[D(x+x_0)], \qquad \psi = B\,\mathrm{sech}^2[D(x+x_0)], \tag{66}$$

which is essentially the solution (41).

5. The coupled ϕ^6 model

This model and the periodic domain walls solutions obtained below are relevant to many structural phase transitions as well as in coupled field theories. In [13] we had considered the following coupled ϕ^6 model, with a bi-quadratic coupling, in one dimension with the potential:

$$V(\phi,\psi) = \left(\frac{a_1}{2}\phi^2 - \frac{b_1}{4}\phi^4 + \frac{c_1}{6}\phi^6\right) + \left(\frac{a_2}{2}\psi^2 - \frac{b_2}{4}\psi^4 + \frac{c_2}{6}\psi^6\right) + \frac{d}{2}\phi^2\psi^2. \tag{67}$$

We now show that in this case we add the following quartic–quadratic and quadratic–quartic coupling terms

$$V' = \frac{e}{4}\phi^4\psi^2 + \frac{f}{2}\phi^2\psi^4, \tag{68}$$

to the potential (67), then in addition to the solutions obtained in [13], there exist truly novel solutions in terms of Lamé polynomials of order 1 and 2 to this coupled problem, even though these are *not* the solutions to the uncoupled ϕ^6 problem. From stability considerations we shall always take $c_1, c_2 > 0$. Further, since we are interested in a model for first-order transition, we shall take $b_1, b_2 > 0$. As far as a_1, a_2 are concerned, their sign is arbitrary and the shape of the potential depends on the ratio $b_1^2/4a_1c_1$ and $b_2^2/4a_2c_2$.

The (static) equations of motion which follow from equations (67) and (68) are

$$\begin{aligned}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} &= a_1\phi - b_1\phi^3 + c_1\phi^5 + \mathrm{d}\phi\psi^2 + e\phi^3\psi^2 + f\phi\psi^4,\\ \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} &= a_2\psi - b_2\psi^3 + c_2\psi^5 + d\psi\phi^2 + \frac{e}{2}\phi^4\psi + f\phi^2\psi^3.\end{aligned} \tag{69}$$

We now show that apart from the solutions discussed in [13], this coupled model also admits rather unusual solutions in terms of the Lamé polynomials of order 1 and 2 which we now discuss one by one. Since there are three Lamé polynomials of order 1 (i.e. sn, cn, dn) and since the field equations are essentially symmetric in ϕ and ψ, we expect six independent solutions to the coupled field equations in terms of Lamé polynomials of order 1. In particular, we first show that there are three periodic bright–bright, two periodic dark–bright and one periodic dark–dark soliton solutions in terms of Lamé polynomials of order 1, which in turn lead to one bright–bright, one dark–dark and one dark–bright hyperbolic soliton solution.

5.1. Solution I

We look for the most general solutions to the coupled equations (69) in terms of the Jacobi elliptic functions $\mathrm{sn}(x, m)$, $\mathrm{cn}(x, m)$ and $\mathrm{dn}(x, m)$ [11]. It is easily shown that

$$\phi = A\,\mathrm{sn}(Dx + x_0, m), \qquad \psi = B\,\mathrm{sn}(Dx + x_0, m), \tag{70}$$

is an exact dark–dark periodic solution to the coupled equations (69) provided the following six coupled equations are satisfied:

$$a_1 = -(1+m)D^2, \tag{71}$$

$$-b_1A^2 + dB^2 = 2mD^2, \tag{72}$$

$$c_1^4 + eA^2B^2 + \frac{f}{2}B^4 = 0, \tag{73}$$

$$a_2 = -(1+m)D^2, \tag{74}$$

$$-b_2B^2 + dA^2 = 2mD^2, \tag{75}$$

$$c_2B^4 + fA^2B^2 + \frac{e}{2}A^4 = 0. \tag{76}$$

We find that the solution exists only if $a_1 < 0, a_2 < 0, e < 0, f < 0$. We obtain

$$D^2 = \frac{|a_1|}{(1+m)}, \qquad A^2 = \frac{2mD^2(d+b_2)}{d^2 - b_1b_2}, \qquad B^2 = \frac{(d+b_1)A^2}{(d+b_2)}, \tag{77}$$

while the three constraints are

$$\begin{aligned} &a_1 = a_2 < 0, \qquad (4c_1c_2 - |e||f|)(d+b_1) = 2(e^2 + 2|f|c_1)(d+b_2), \\ &(4c_1c_2 - |e||f|)^2 = 4(e^2 + 2|f|c_1)(f^2 + 2|e|c_2). \end{aligned} \tag{78}$$

In the limit of $m = 1$, the periodic solution (70) goes over to the hyperbolic dark–dark soliton solution

$$\phi = A\tanh(Dx + x_0), \qquad \psi = B\tanh(Dx + x_0), \tag{79}$$

provided the constraints (77) and (78) with $m = 1$ are satisfied.

5.2. Solution II

It is easy to show that

$$\phi = A\,\mathrm{cn}(Dx + x_0, m), \qquad \psi = B\,\mathrm{cn}(Dx + x_0, m) \tag{80}$$

is an exact bright–bright periodic solution to the coupled equations (69) provided six coupled equations similar to equations (71)–(76) are satisfied. In particular, we obtain

$$D^2 = \frac{a_1}{(2m-1)}, \qquad a_1 = a_2, \qquad A^2 = \frac{2m(d+b_2)D^2}{(b_1b_2 - d^2)}, \qquad B^2 = \frac{(b_1+d)A^2}{(b_2+d)}, \tag{81}$$

while the remaining two constraints are again given by equation (78). Note that $a_1 = a_2 > (<)\ 0$ if $m > (<)\ 1/2$.

In the limit of $m = 1$, the periodic solution (80) goes over to the hyperbolic bright–bright solution

$$\phi = A\,\mathrm{sech}(Dx + x_0), \qquad \psi = B\,\mathrm{sech}(Dx + x_0), \tag{82}$$

provided the constraints (78) and (81) with $m = 1$ are satisfied.

5.3. *Solution III*

Yet another bright–bright periodic soliton solution is

$$\phi = A\,\mathrm{dn}(Dx + x_0, m), \qquad \psi = B\,\mathrm{dn}(Dx + x_0, m), \tag{83}$$

provided six coupled equations similar to equations (71)–(76) are satisfied. In particular, we obtain

$$D^2 = \frac{a_1}{(2-m)}, \qquad a_1 = a_2 > 0, \qquad A^2 = \frac{2(d+b_2)D^2}{(b_1b_2 - d^2)}, \qquad B^2 = \frac{(b_1+d)A^2}{(b_2+d)}, \tag{84}$$

while the remaining two constraints are again given by equation (78).

In the limit of $m = 1$, the periodic solution (83) again goes over to the hyperbolic bright–bright soliton solution (82).

Note that while for the solution (70), $d^2 > b_1b_2$, for the solutions (80) and (83), its the other way around, i.e. $d^2 < b_1b_2$.

5.4. *Solution IV*

Yet another bright–bright periodic soliton solution is

$$\phi = A\sqrt{m}\,\mathrm{cn}(Dx + x_0, m), \qquad \psi = B\,\mathrm{dn}(Dx + x_0, m), \tag{85}$$

provided six coupled equations similar to equations (71)–(76) are satisfied. We find that this solution is also valid only if $e < 0$, $f < 0$. Further, two of the relations are given by

$$(4c_1c_2 - |e||f|)^2 = 4(e^2 + 2|f|c_1)(f^2 + 2|e|c_2), \qquad (4c_1c_2 - |e||f|)A^2 = 2(f^2 + 2|e|c_2)B^2. \tag{86}$$

In the limit of $m = 1$, the periodic solution (85) again goes over to the hyperbolic bright–bright soliton solution (82).

5.5. *Solution V*

In addition, there are two dark–bright periodic soliton solutions. One of them is

$$\phi = A\,\mathrm{sn}(Dx + x_0, m), \qquad \psi = B\,\mathrm{cn}(Dx + x_0, m), \tag{87}$$

provided six coupled equations similar to equations (71)–(76) are satisfied. We find that this solution exists provided

$$A^2 = \frac{b_1 \pm \sqrt{b_1^2 - 4a_1c_1 - 4(1-m)D^2c_1}}{2c_1}, \qquad B^2 = \frac{b_2 \pm \sqrt{b_2^2 - 4a_2c_2 - 4D^2c_2}}{2c_2}. \tag{88}$$

Further, unlike the previous four solutions, this solution exists only if $e > 0$, $f > 0$ and two of the constraints are given by

$$(4c_1c_2 - ef)A^2 = 2(2ec_2 - f^2)B^2, \qquad (4c_1c_2 - ef)^2 = 4(2ec_2 - f^2)(2fc_1 - e^2), \tag{89}$$

while the other two constraints are

$$B^2 = \frac{b_1A^2 - 2a_1 - 2D^2}{d + eA^2} = \frac{2a_2 + 2(1-m)D^2 + dA^2}{b_2 - fA^2}. \tag{90}$$

In the limit of $m = 1$, the periodic solution (87) goes over to the hyperbolic dark–bright soliton solution

$$\phi = A\tanh(Dx + x_0), \qquad \psi = B\,\mathrm{sech}(Dx + x_0), \tag{91}$$

satisfying the constraints (88)–(90) with $m = 1$.

5.6. Solution VI

Another dark–bright periodic soliton solution is given by

$$\phi = A\sqrt{m}\,\mathrm{sn}(Dx + x_0, m), \qquad \psi = B\,\mathrm{dn}(Dx + x_0, m), \tag{92}$$

provided six coupled equations similar to equations (71)–(76) are satisfied. We find that this solution exists provided two of the constraints are again given by equation (89) while A^2 and B^2 are now given by

$$A^2 = \frac{b_1 \pm \sqrt{b_1^2 - 4a_1c_1 + 4(1-m)D^2c_1}}{2c_1}, \qquad B^2 = \frac{b_2 \pm \sqrt{b_2^2 - 4a_2c_2 - 4mD^2c_2}}{2c_2}, \tag{93}$$

$$B^2 = \frac{b_1A^2 - 2a_1 - 2mD^2}{d + eA^2} = \frac{2a_2 - 2(1-m)D^2 + dA^2}{b_2 - fA^2}. \tag{94}$$

In the limit of $m = 1$, the periodic solution (92) goes over to the hyperbolic dark–bright soliton solution (91).

We now show that quite remarkably, the coupled model characterized by the field equations (69) not only admits periodic solutions in terms of Lamé polynomials of order 1, but it also admits novel periodic solutions in terms of Lamé polynomials of order 2. It is worth recalling once again that neither Lamé polynomials of order 1 nor of order 2 are solutions of the uncoupled ϕ^6 problem. Since there are five Lamé polynomials of order 2, and since two of these are of the form $A\,\mathrm{sn}^2[D(x + x_0), m] + F$, and further, the two field equations (69) are symmetrical in ϕ and ψ, in principle there could be ten solutions of order 2. However, it turns out that only two of these are admitted by the field equations (69) which we now discuss.

5.7. Solution VII

It is easily shown that

$$\phi = A\,\mathrm{sn}^2(Dx + x_0, m) + F, \qquad \psi = B\,\mathrm{sn}(Dx + x_0, m)\mathrm{cn}(Dx + x_0, m), \tag{95}$$

is an exact periodic solution to the coupled equations (69) provided the following 11 coupled equations are satisfied:

$$a_1F - b_1F^3 + c_1F^5 = 2AD^2, \tag{96}$$

$$a_1A - 3b_1AF^2 + 5c_1AF^4 + eB^2F^3 + dB^2F = -4(1+m)AD^2, \tag{97}$$

$$-3b_1A^2F + 10c_1A^2F^3 + eB^2F^2(3A - F) + \frac{f}{2}B^4F + dB^2(A - F) = 6mAD^2, \tag{98}$$

$$-b_1A^3 + 10c_1A^3F^2 + 3eAB^2F(A - F) + \frac{f}{2}B^4(A - 2F) - dAB^2 = 0, \tag{99}$$

$$5c_1A^4F + eA^2B^2(A - 3F) + \frac{f}{2}B^4(F - 2A) = 0, \tag{100}$$

$$c_1A^4 - eA^2B^2 + \frac{f}{2}B^4 = 0. \tag{101}$$

$$a_2 + \frac{e}{2}F^4 + dF^2 = -(4+m)D^2, \tag{102}$$

$$-b_2B^2 + 2eAF^3 + fF^2B^2 + 2dAF = 6mD^2, \tag{103}$$

$$b_2B^2 + c_2B^4 + 3eA^2F^2 + fB^2F(2A - F) + dA^2 = 0, \tag{104}$$

$$-2c_2B^4 + 2eA^3F + fAB^2(A - 2F) = 0, \tag{105}$$

$$c_2B^4 - fA^2B^2 + \frac{e}{2}A^4 = 0. \tag{106}$$

Four of these equations determine the four unknowns A, B, D, F while the other equations give constraints between the nine parameters $a_{1,2}, b_{1,2}, c_{1,2}, d, e, f$. In particular, we find that the solution exists only if

$$eA^2 = fB^2, \qquad e^3 = 8c_1^2c_2, \qquad f^3 = 8c_2^2c_1, \tag{107}$$

and further if $F \neq 0$.

In the limit of $m = 1$, the periodic solution (95) goes over to the hyperbolic solution

$$\phi = A\tanh^2(Dx + x_0) + F, \qquad \psi = B\tanh(Dx + x_0)\text{sech}(Dx + x_0), \tag{108}$$

provided the constraints (96)–(106) with $m = 1$ are satisfied. There is one special case when this solution takes a simpler form, i.e. when $A = -F$, the solution is given by

$$\phi = -A\,\text{sech}^2(Dx + x_0), \qquad \psi = B\tanh(Dx + x_0)\text{sech}(Dx + x_0), \tag{109}$$

provided equation (107) is satisfied and further

$$D^2 = \frac{a_1}{4}, \qquad a_1 = 4a_2 > 0, \qquad d = -b_2, \qquad b_2^2(f - e) = 6a_2c_2e,$$
$$B^2 = \frac{6a_2}{b_2}, \qquad A^2 = \frac{-b_1 \pm \sqrt{b_1^2 - 6a_1c_1}}{2c_1}. \tag{110}$$

Thus in the ϕ variable, one is at $T < T_c$ (transition temperature) since $b_1^2 > 6a_1c_1$.

5.8. Solution VIII

The other allowed solution is

$$\phi = A\,\text{sn}^2(Dx + x_0, m) + F, \qquad \psi = B\,\text{sn}(Dx + x_0, m)\,\text{dn}(Dx + x_0, m), \tag{111}$$

which is an exact periodic solution to the coupled equations (69) provided 11 coupled equations similar to equations (96)–(106) are satisfied. We find that the solution exists only if

$$emA^2 = fB^2, \qquad e^3 = 8c_1^2c_2, \qquad f^3 = 8c_2^2c_1, \tag{112}$$

and further if $F \neq 0$.

In the limit of $m = 1$, the periodic solution (111) also goes over to the hyperbolic solution (108).

6. Conclusions

In this paper, we have shown that the Lamé polynomials of order 2 are the periodic solutions of the coupled ϕ^4 problems when either the potentials in both the fields are symmetric, or when both are asymmetric or when the potential is symmetric in one and asymmetric in the other field. The latter model is also relevant for reconstructive phase transitions in many materials [14, 15]. These are novel solutions in the sense that while they are solutions of the coupled problems, they are not the solutions of the corresponding uncoupled problems. In particular, in all the three cases, we have shown that while the Lamé polynomials of *order 1* are the solutions of *both* the coupled and the uncoupled problems, the Lamé polynomials of *order 2* are the solutions of the coupled problems, but *not* of the uncoupled ones. A physical realization of these solutions is periodic domain walls in both spin orientations and lattice strain, e.g. in multiferroic materials.

It is worth emphasizing here that there are three Lamé polynomials of order 1 and as a result one has nine different solutions for the coupled ϕ^4 problems which we have presented in [9, 10]. Since there are five Lamé polynomials of order 2 one would have naively expected 16 solutions of order 2 for the coupled ϕ^4 problems (note that two of the Lamé polynomials are of the form $F + A\,\mathrm{sn}^2(x, m)$). However, it turned out that while there are *six* allowed solutions in the symmetric ϕ^4 case in an external field, only *one* solution is allowed in the asymmetric case and *four* solutions are allowed in the asymmetric–symmetric case.

It may be noted here that previously, Hioe and Salter [18] had shown similar features for coupled nonlinear Schrödinger (NLS) equations. They also pointed out that precisely when such solutions exist, the coupled NLS equations are integrable and they pass the Painlevé test [19]. Thus one might get the impression that the existence of higher order Lamé polynomials as solutions of the coupled problem (but not that of the uncoupled problem) could be related to the integrability of the coupled as well as the uncoupled systems. However, our work has clearly shown that this is not so. In particular, it is well known that the ϕ^4 field theory (both symmetric or asymmetric or mixed) is a nonintegrable field theory. As a further support to our argument, we considered in section 5 and elsewhere [17] a coupled ϕ^6 model studied by us recently [13] and show that provided we add extra interaction terms which are quadratic–quartic in the two fields, then Lamé polynomials of order 2 are also the solutions of the coupled problem (though they are not the solutions of the uncoupled problem).

Based on these examples, we conjecture that for most of the coupled models, novel solutions (i.e. those which are solutions of the coupled but not the uncoupled problem) will exist as long as there is a coupling term between the fields which is of the same order as the highest power of the uncoupled fields. Further, in those cases where Lamé polynomials, of say order 1, are solutions of the uncoupled problem, we conjecture that if there are n-coupled fields with coupling terms being of the same order as the highest power of the uncoupled fields, then Lamé polynomials of order n will also be the solutions of the coupled problem. Note that four coupled ϕ^4 fields are required to model different magnetic phases of the hexagonal multiferroic materials [20]. It will be interesting to examine our conjecture in a few coupled field theory models. We hope to address these issues in future studies.

In this paper, we have shown that the Lamé polynomials of order 1 and 2 are periodic solutions of a coupled ϕ^6 problem. These are novel solutions in the sense that while they are solutions of the coupled ϕ^6 problem, they are not the solutions of the corresponding uncoupled problems. In particular, we have obtained six solutions in terms of Lamé polynomials of order 1 and two solutions in terms of Lamé polynomials of order 2. These results are applicable to both the structural phase transitions [4, 5] and field theoretic contexts [6–8] and correspond to periodic domain walls in crystal structures with different symmetry or in two different fields.

Although not presented here, we have also obtained four solutions of the coupled ϕ^6–ϕ^4 problem in [17], both when the ϕ^4 potential corresponds to a first order (i.e. asymmetric) as well as a second-order transition (i.e. symmetric). Note that while the solutions of the coupled problem are also the solutions of the uncoupled ϕ^6 problem, but they are not the solutions of either the symmetric or the asymmetric uncoupled ϕ^4 problems. These solutions are also useful in understanding coexistence of different crystalline structures in elements [15, 21, 22] and ferroelectrics [23, 24]. It will be interesting to obtain solutions of a few other coupled field theories and with couplings that are not bi-quadratic. An example of a coupled model with linear–quadratic coupling occurs in the context of isostructural transitions [25]. It is conceivable that in some special cases a linear–cubic coupling may be symmetry allowed.

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